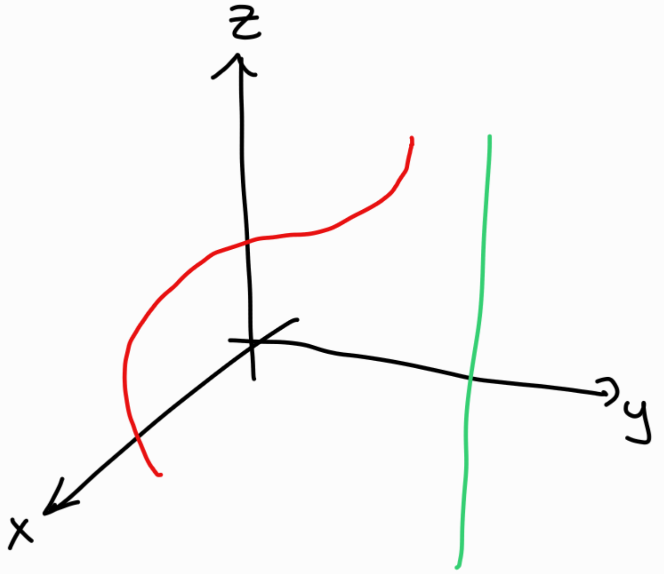
What is a "curve"?



A "continuous, 1-dimensional stream" of points

Think of it as the trajectory of a moving particle.

Q. How to describe a curve?

A. Describe the movement of a particle.

Given time t, the particle's location is $(a(t), b(t), c(t))$

$\Rightarrow$ Each $a(t)$, $b(t)$, $c(t)$ is a function of t,

and the equations $\left\{ \begin{array}{l} x = a(t) \\ y = b(t) \\ z = c(t) \end{array} \right\}$ describe a curve.

A curve expressed in this way (by introducing extra time variable t)

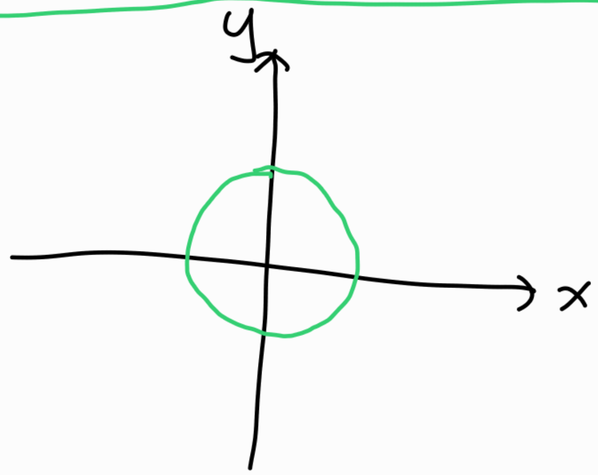
is called a parametric curve.

$$\left.\begin{cases} x = a(t) \\ y = b(t) \\ z = c(t) \end{cases}\right\} \leftarrow \text{parametric equation of the curve}$$

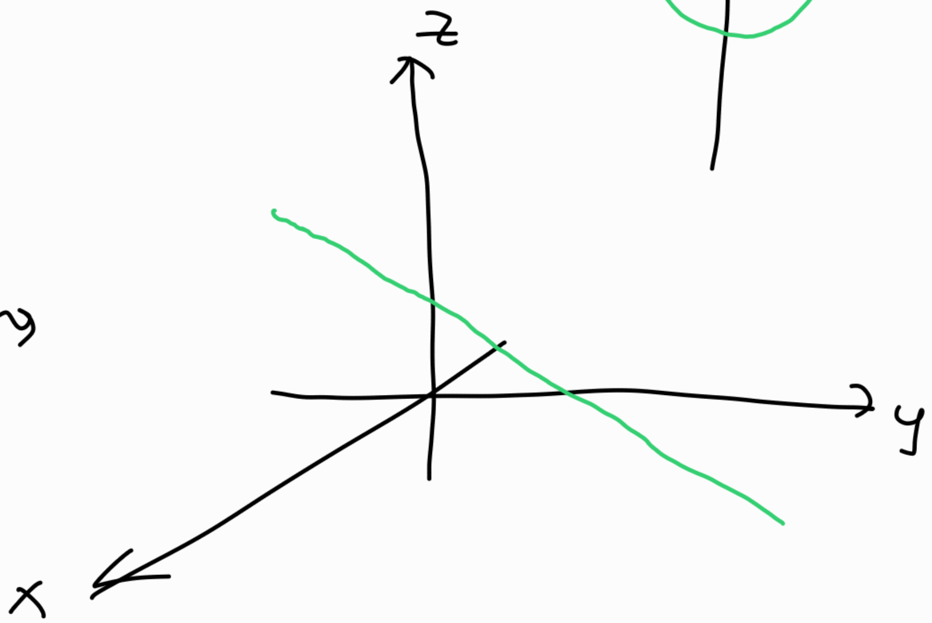
(can do similarly in 2D)

Example

$x = \cos(t)$
$y = \sin(t)$ $\rightsquigarrow$



$x = t$
$y = 2t$ $\rightsquigarrow$
$z = 1$



Remark

Parametric equations

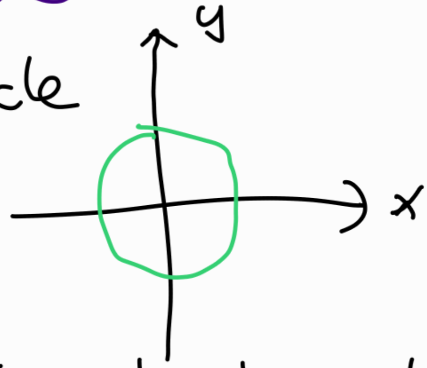
Pro: Easy to deal with mathematically

Con: Different equations may represent the same curve.

We describe a curve by describe a moving object along it, but there are many ways to move along a given path!

Trip ≠ Path

Example Given the same circle

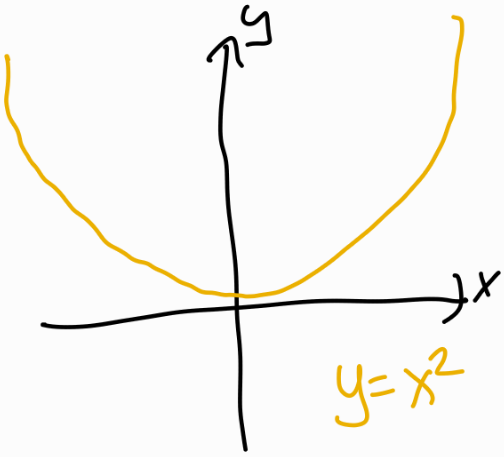


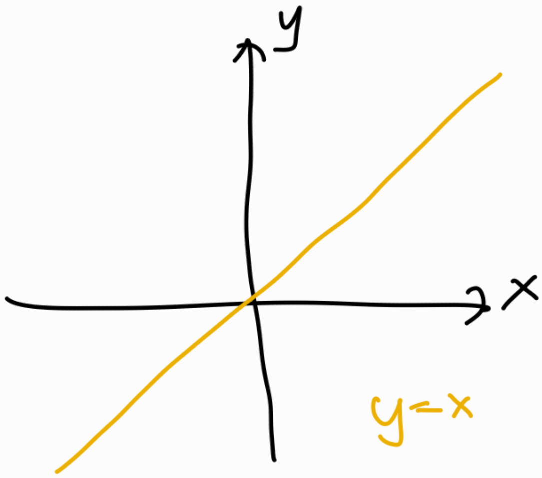
there are many ways to travel along it:

$x=\cos(t)$ $y=\sin(t)$ Trip #1	$x=\cos(2t)$ $y=\sin(2t)$ Trip #2 x2 faster	$x=\cos(-t)$ $y=\sin(-t)$ Trip #3 Reverse trip	$x=\cos(3t^3+t)$ $y=\sin(3t^3+t)$ Trip #4 ???

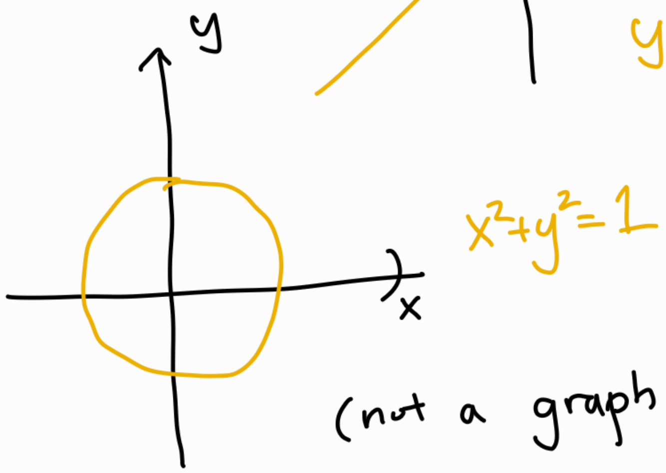
Later, we will learn how to "standardize" a trip.

A 2D graph $y=f(x)$ is also a curve.





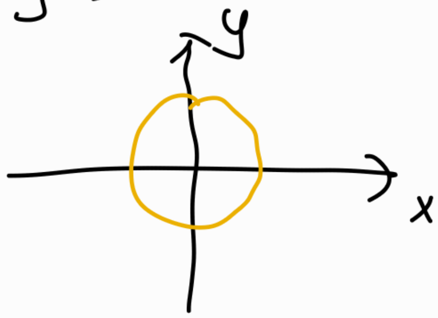
Something more:



(not a graph, but still a curve).

Equations describing a curve without introducing extra time variable are called implicit equations.

Example From parametric equations $x = \cos(t)$, $y = \sin(t)$ of the circle of radius 1

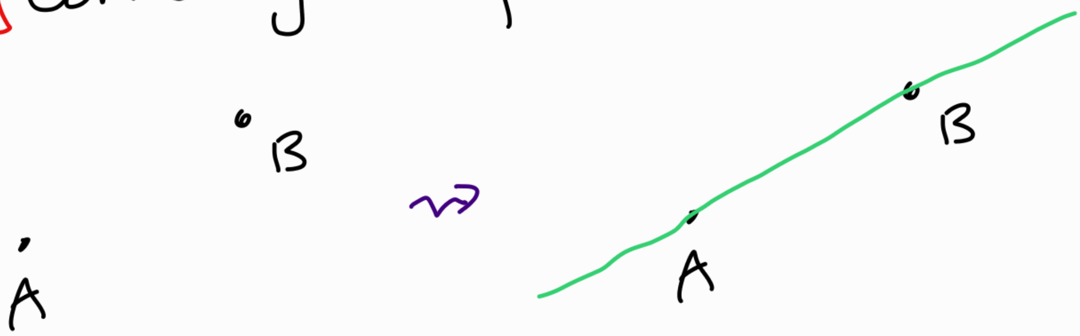


$\rightsquigarrow$ $x^2 + y^2 = 1$, because $\cos^2 t + \sin^2 t = 1$.

Easiest curves: lines

Two ways to think about a line:

$\boxed{\text{I.}}$ Connecting two points A, B



$\boxed{\text{II}}$ Starting from point A, extend along a vector $\vec{v}$.

$\boxed{\text{I}} \rightsquigarrow \boxed{\text{II}}$: Take $\vec{v} = \overrightarrow{AB}$

Line connecting A & B
= Line passing thru A, w/ direction $\overrightarrow{AB}$.

$\boxed{\text{II}} \rightsquigarrow$ parametric equations :

$A = (a_1, a_2, a_3)$ $\quad \vec{v} = \langle v_1, v_2, v_3 \rangle$

then the line L passing thru A w/ direction $\vec{v}$

is

$$\boxed{\begin{aligned} L:\ x &= v_1 t + a_1 \\ y &= v_2 t + a_2 \\ z &= v_3 t + a_3 \end{aligned}}$$

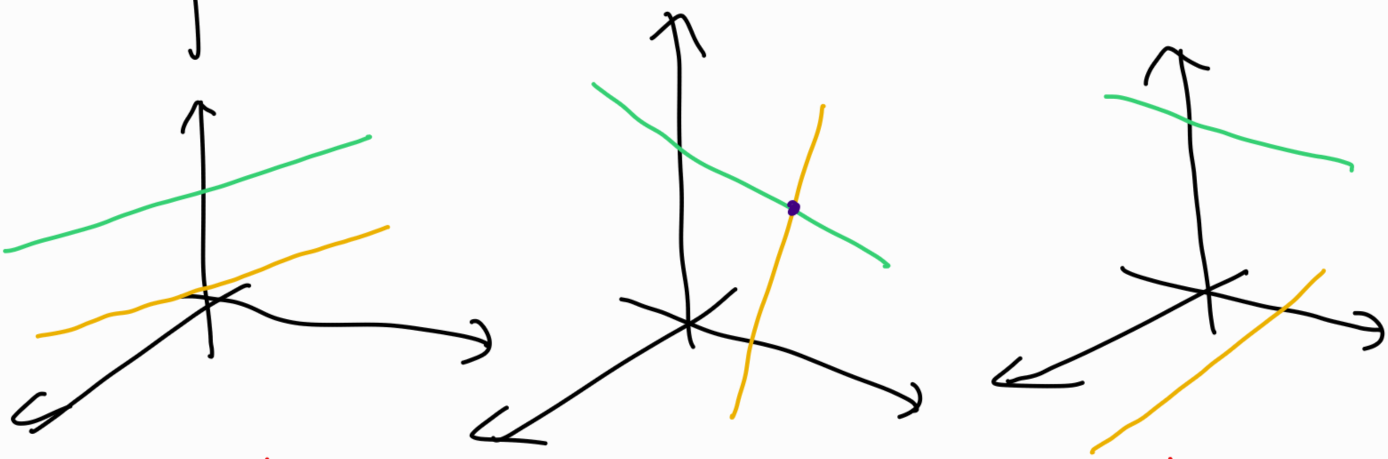
We call A a reference point and $\vec{v}$ a directional vector.

Remark · Can replace directional vector $\vec{v}$ with any parallel vector.

Line passing through A w/ direction $\vec{v}$
= Line passing through A w/ direction $r\vec{v}$ for any scalar r.

· Also can replace reference point with any point on the same line.

Given two lines L_1, L_2 in 3D, there are three possibilities:



Parallel | Intersecting | Skew (Neither parallel nor intersecting)

How to distinguish

Assume L_1 has reference point $A = (a_1, a_2, a_3)$
directional vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$

L_2 has reference point $B = (b_1, b_2, b_3)$
directional vector $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

Angle between L_1 & L_2 = Acute angle between $\vec{v}$ & $\vec{w}$.

Already learned how to check, check if $\frac{v_1}{w_1} = \frac{v_2}{w_2} = \frac{v_3}{w_3}$.

Parallel: if directional vectors $\vec{v_1}, \vec{v_2}$ are parallel.

Intersecting: if there is a point of intersection between L_1, L_2

Skew: neither parallel nor intersecting

Skew: neither parallel nor intersecting

↑ Try both above and conclude

How to find a possible point of intersection:

Write parametric equations for L_1, L_2 using different time variables t and s:

$$L_1: \begin{cases} x = v_1 t + a_1 \\ y = v_2 t + a_2 \\ z = v_3 t + a_3 \end{cases} \qquad L_2: \begin{cases} x = w_1 s + b_1 \\ y = w_2 s + b_2 \\ z = w_3 s + b_3 \end{cases}$$

(Two moving objects are unrelated, so different sense of time)

Then, L_1 and L_2 intersects if a system of linear equations

$$\left[\begin{aligned} v_1 t + a_1 &= w_1 s + b_1 \\ v_2 t + a_2 &= w_2 s + b_2 \\ v_3 t + a_3 &= w_3 s + b_3 \end{aligned}\right]$$

has a solution in (t, s). If so, the point of intersection is the point $(v_1 t + a_1, v_2 t + a_2, v_3 t + a_3)$
$= (w_1 s + b_1, w_2 s + b_2, w_3 s + b_3)$

<u>Example</u> L_1: reference point $(1,1,0)$
directional vector $\langle 0, 0, -1 \rangle$

L_2: reference point $(2,3,4)$

directional vector $\langle 1,2,3 \rangle$

Are L_1, L_2 parallel / intersecting / skew?

If intersecting, find their point of intersection.

<u>Solution</u>. NOT PARALLEL, because $\langle 0,0,-1 \rangle$ & $\langle 1,2,3 \rangle$ are not parallel. $\frac{0}{1} = \frac{0}{2} \neq \frac{-1}{3}$.

Do they intersect? To see this, let's solve a system of linear equations

$$\begin{bmatrix} 0 \cdot t + 1 & = 1 \cdot s + 2 \\ 0 \cdot t + 1 & = 2 \cdot s + 3 \\ -1 \cdot t + 0 & = 3 \cdot s + 4 \end{bmatrix}$$

Simplify $\rightarrow$
$$\begin{bmatrix} 1 = s + 2 & \cdots & (A) \\ 1 = 2s + 3 & \cdots & (B) \\ -t = 3s + 4 & \cdots & (C) \end{bmatrix}$$

(A) $\rightsquigarrow$ $s = -1$

match ✓

(B) $\rightsquigarrow$ $2s = -2 \rightsquigarrow s = -1$

(C) $\rightsquigarrow$ $t = -3s - 4 = -3 \cdot (-1) - 4 = 3 - 4 = -1$.

$\Rightarrow$ $(t, s) = (-1, -1)$ solves the system,

so INTERSECT.

point of intersection $= (1, 1, -t) = (1, 1, 1)$.

point of intersection